# Approximate KKT Points:A Theoretical & Numerical Study

May 13, 2010

## 1 Introduction

The Karush-Kuhn-Tucker(KKT) conditions are necessary for a solution in a non-linear programming problem to be optimal, provided some regularity conditions are satisfied, and play an important part in optimization theory[14]. However, the KKT conditions have not been widely used in optimization algorithm design, primarily owing to the fact that these are point conditions. Often, commercial softwares like KNITRO[2] and MATLAB Optimization toolbox[5], use KKT conditions as a stopping criterion. However, these are never used in the algorithm design. If however, one could judge the proximity and direction of the optima using some metric derived from the KKT condition violations, this could be very helpful for algorithmic design point of view. This study is an aim to satisfy the former of the requirements. A KKT proximity metric has been derived to indicate the closeness to the optima of the given iterate.

The structure of the report is as follows. Section 2 states the KKT conditions for smooth problems. It is noted in 3 that KKT conditions are point conditions if complimentary slackness is preserved. This simple example illustrates the difficulties in using KKT error are a proximity measure since points very close to the optima may have very high KKT error. Section 4 presents theoretical ideas and results concerning *Approximate KKT Error*, while 5 explores the use of KKT proximity measure as termination criterion in commercial software. Simulation results 6 for the proposed relaxation scheme are presented, using generation outputs from Real Coded Genetic Algorithms, manually appending the optima whereever the GA doesn't converge. Conclusions and further research work are suggested in 7.

## 2 KKT Conditions: Smooth & Nonsmooth Cases

### 2.1 Smooth Case

For the given single-objective, constrained smooth optimization problem:

$$\begin{aligned} \text{Minimize} \quad & f(\mathbf{x}), \\ \text{Subject to} \quad & g_j(\mathbf{x}) \leq 0, \quad j = 1, 2, \ldots, m, \end{aligned} \tag{1}$$

the Karush-Kuhn-Tucker (KKT) optimality conditions are given as follows:

$$\nabla f(\bar{\mathbf{x}}) + \sum_{j=1}^{m} u_j \nabla g_j(\bar{\mathbf{x}}) = \mathbf{0}, \tag{2}$$

$$g_j(\bar{\mathbf{x}}) \leq 0, \quad \forall j, \tag{3}$$

$$u_j g_j(\bar{\mathbf{x}}) = 0, \quad \forall j, \tag{4}$$

$$u_j \geq 0, \quad \forall j. \tag{5}$$

The parameter $u_j$ is called the Lagrange multiplier for the $j$-th constraint. Any solution $\bar{\mathbf{x}}$ that satisfies all the above conditions is called a KKT point [14]. Equation 2 is called the *equilibrium equation* and a norm of the left side vector is called the KKT error. Equation 4 for every constraint is called the *complimentary slackness* equation. Note that the conditions given in 3 ensure feasibility for $\bar{\mathbf{x}}$ while the equation 5 arises due to the minimization nature of the problem given in equation 1.

The complimentary slackness condition implies that if a KKT point $\bar{\mathbf{x}}$ makes a constraint inactive (say, $g_j(\bar{\mathbf{x}}) < 0$), the corresponding Lagrange multiplier $u_j$ is zero. On the other hand, along with equation 5, we conclude that if the KKT point makes the $j$-th constraint active (meaning $g_j(\bar{\mathbf{x}}) = 0$), $u_j$ may take any non-negative values. Thus, the equilibrium equation requires that the negative of the gradient vector of the objective function at the KKT point be a positive linear combination of the gradient vectors of the active constraints.

It is important to note that a KKT point is not necessarily a minima of the original problem. Further conditions (in general, involving constraint qualification conditions or second-order derivatives) are necessary to establish the optimality of a point. However, in this paper, we keep our discussions to KKT points which are candidates for the minimum point.

The KKT conditions clearly state that the KKT error ($\|\nabla f(\bar{\mathbf{x}}) + \sum_{j=1}^{m} u_j \nabla g_j(\bar{\mathbf{x}})\|$) is zero at the KKT point. However, it is not clear and is not adequately mentioned in textbooks on mathematical optimization as to how the KKT error varies for points even in the proximity of the KKT point. If the KKT error reduces monotonically for points as we get closer to the KKT point, the KKT error can be reliably used as a *termination criterion* for any constrained optimization algorithm, including an evolutionary algorithm. We investigate this aspect in the following subsection.

# 3 KKT Error with Complimentary Slackness in the Smooth Case

For a feasible point $\mathbf{x}^k$, we suggest a simple scheme for computing the KKT error using the complementary slackness condition. We only consider all constraints which are active with the following constraint index set: $I = \{j | g_j(\mathbf{x}^k) = 0\}$. Now, we solve the following optimization problem to find the Lagrange multiplier for all active constraints:

$$\begin{array}{rl} \text{Minimize} & \|\nabla f(\mathbf{x}^k) + \sum_{j\in I} u_j \nabla g_j(\mathbf{x}^k)\|, \\ \text{Subject to} & u_j \geq 0 \quad \forall j \in I. \end{array} \tag{6}$$

Note that only $u_j$'s for active constraints are variables to the above problem. The objective of the optimal solution of the above problem is the KKT error at $\mathbf{x}^k$. We illustrate the difficulties with this approach by applying it on a two-variable problem.

Consider the problem (Figure 1):

$$\begin{aligned}
\text{Minimize} \quad f(\mathbf{x}) &= x^2 + y^2 - 10x + 4y + 2, \\
\text{Subject to} \quad g_1(\mathbf{x}) &= x^2 + y - 6 \leq 0, \\
g_2(\mathbf{x}) &= x - y \leq 0, \\
g_3(\mathbf{x}) &= -x \leq 0.
\end{aligned}$$

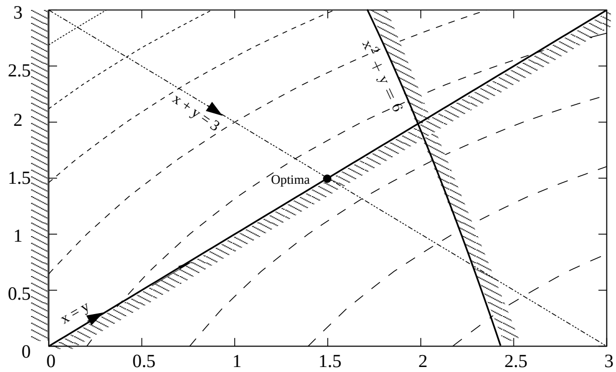


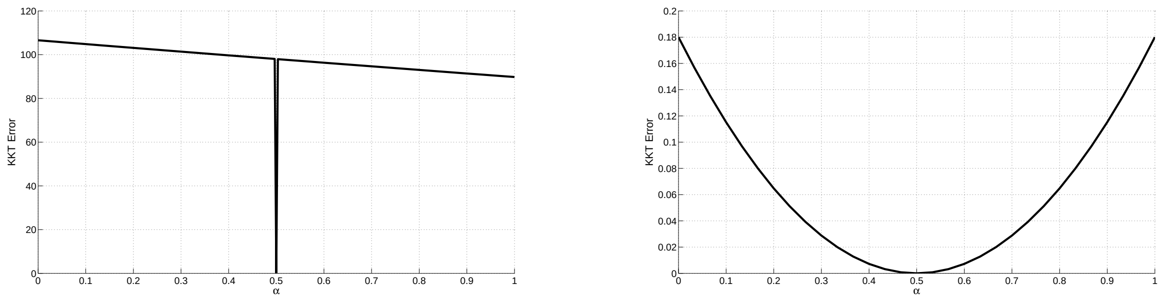

Figure 1: Contour plot of the objective and feasible region.

It can be verified that the point $\mathbf{x}^* = (1.5, 1.5)^T$ is the global optimum (also a KKT point since $f$ and $g_i$ are convex). At $\mathbf{x}^*$, the second constraint is active. Now, consider a point close to the optimum, say $(1.49, 1.51)^T$. At this feasible point, none of the constraints are active, and the KKT error is simply $\|\nabla f\|$ which is not equal to zero. Thus, the KKT error with strict complementary slackness does not give us any information on the proximity from the optimum. On this account, the KKT error computed from the complementary slackness condition cannot be used as a reliable termination criterion for an optimization algorithm.

Further, consider sequences of points approaching the optimum along two paths, $x = y$ and $x + y = 3$, with their KKT error plots shown in Figures 2(a) and 2(b), respectively. On the path $x = y$, the constraint given by equation 7 is active and the error smoothly reduces to zero. However, on the path $x + y = 3$, all constraints are inactive and we can observe the discontinuous behavior of the error, with it being zero at $\mathbf{x}^*$.

The above results show that for some problems, the KKT error value computed using the complementary slackness condition depends on the manner the iterates approach the KKT point. For certain series of iterates, the KKT error can remain quite high in the vicinity of the optimum before suddenly dropping to zero at the exact KKT point or at the optimum. The results also indicate that there is no particular relationship which can be obtained between the KKT error and the closeness of a solution to the optimum.

If we observe carefully, this happens mainly due to the discontinuity enforced by the complimentary slackness condition. For a feasible point very near to a constraint boundary, but not on the boundary, $u_j$ must be zero whereas as soon as the point is on the constraint boundary, $u_j$ is




(a) The KKT-Error reduces smoothly to zero ( at $x^*$ while traversing along $x = y = 1.35 + 0.3\alpha$ keeping the constraint active.

(b) The KKT-Error has a discontinuity at $x^*$, where it is zero. Plot of points along the line $x = 1.35 + 0.3\alpha, y = 1.65 - 0.3\alpha$

Figure 2: Behaviour of KKT-Error

relaxed to take any non-negative value. Hence, the participation of the gradient vector of the corresponding constraint in the equilibrium equation may be sudden, thereby causing a fluctuation in the KKT error value.

Thus, in order to use the KKT error as a KKT-proximity measure or as a termination criterion for an algorith so that the progress of the algorithm towards the optimum solution can be measured, we need to relax the complimentary slackness condition. In the following section , we discuss a couple of such recent efforts.

However, Deb et al.[3] suggested a KKT based techinique for establishing KKT-optimality conditions for multi-objective optimization problems. Since KKT conditions for multi-objective optimization involve additional Lagrange parameters related to objective functions, the optimization task involved in computing the KKT error has a greater flexibility in reducing its value. However, in handling single-objective optimization, there is no additional Lagrange parameter for the objective and only flexibility comes from the active constraints. We also suggest a relaxation technique of active constraints in the following subsection.

# 4 Approximate KKT Optimality Conditions

In this section we aim to study the approximate KKT optimality conditions and their relationship with the exact KKT optimality conditions. Very recently Andreani et. al.[8] had introduced same notions of Approximate KKT conditions. Andreani et. al.[8] studied only the smooth case while we consider both the smooth and non-smooth cases. We would like to point out that our approach to approximate KKT points is quiet different from that of Andreani et. al.[8].

## 4.1 $\epsilon$-KKT conditions: Smooth Case

We will concentrate on the following simple optimization problem (P):

$$\begin{array}{ll} \text{Min} & f(\mathbf{x}) \\ \text{subject to} & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \ldots, m. \end{array} \tag{7}$$

We will assume that $f$ and each $g_i$ $(i = 1, 2, \ldots, m)$ are smooth functions. Our main aim in this section is to define certain notions of approximate KKT points and show that, if a sequence of such points converge to a point where some constraint qualification is also satisfied, then such a point is a KKT point.

**Definition 1** *A point $\bar{\mathbf{x}}$ which is feasible to (P) is said to be an $\epsilon$-KKT point if given $\epsilon > 0$, there exist scalars $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$ such that*

1. $\|\nabla f(\bar{\mathbf{x}}) + \sum_{i=1}^m \lambda_i \nabla g_i(\bar{\mathbf{x}})\| \leq \epsilon$,
2. $\lambda_i g_i(\bar{\mathbf{x}}) = 0$, *for* $i = 1, 2, \ldots, m$

In order to state our main result we need the notion of Mangasarian-Fromovitz constraint qualification (MFCQ) which is given as follows: The constraints of the problem (P) is said to satisfy MFCQ at $\bar{x}$ which is feasible if there exists a vector $d \in \mathbf{R}^n$ such that $\langle \nabla g_i(\bar{\mathbf{x}}), d\rangle < 0$, $\forall i \in I(\bar{\mathbf{x}})$, where $I(\bar{\mathbf{x}}) = \{i : g_i(\bar{\mathbf{x}}) = 0\}$ is the set of active constraints.

The MFCQ can be alternatively stated in the following equivalent form which can be deduced using separation theorem for convex sets.

The constraints of (P) satisfies MFCQ at a feasible $\bar{\mathbf{x}}$ if there exists no vector $\lambda \neq 0$, $\lambda \in \mathbf{R}^m_+$, with $\lambda_i \geq 0$, $i \in I(\bar{\mathbf{x}})$ and $\lambda_i = 0$ for $i \neq I(\bar{\mathbf{x}})$ with

$$\sum_{i=1}^m \lambda_i \nabla g_i(\bar{\mathbf{x}}) = 0.$$

Now we will state our main result in the smooth case.

**Theorem 1** *Let $\{\mathbf{x}^\nu\}$ be a sequence of feasible points of (P) such that $\mathbf{x}^\nu \to \bar{\mathbf{x}}$ as $\nu \to \infty$. Let $\{\epsilon_\nu\}$ be a sequence of positive real numbers such that $\epsilon_\nu \downarrow 0$, as $\nu \to \infty$. Further assume that for each $\nu$, $\mathbf{x}^\nu$ is an $\epsilon_\nu$-KKT point of (P). If MFCQ holds at $\bar{\mathbf{x}}$, then $\bar{\mathbf{x}}$ is a KKT point.*

**Proof** Since $\mathbf{x}^\nu$ is an $\epsilon_\nu$-KKT point for (P), it is clear from the definition that $\mathbf{x}^\nu \in C$ for each $\nu$ and as each $g_i$ is continuous and $\{\mathbf{x}^\nu\} \to \bar{\mathbf{x}}$ it is clear that $\bar{\mathbf{x}}$ is a feasible point for (P). Now from the definition of $\epsilon_\nu$-KKT points we have for each $\nu$, there exists a vector $\lambda^\nu \in \mathbf{R}^m_+$ such that

1. $\|\nabla f(\mathbf{x}^\nu) + \sum_{i=1}^m \lambda_i^\nu \nabla g_i(\mathbf{x}^\nu)\| \leq \epsilon_\nu$,
2. $\lambda_i^\nu g_i(\bar{\mathbf{x}^\nu}) = 0$, for $i = 1, 2, \ldots, m$.

Our claim is that the sequence $\lambda^\nu$ is bounded. On the contrary assume that $\lambda^\nu$ is not bounded. Hence, $\|\lambda^\nu\| \to \infty$ as $\nu \to \infty$. Now consider the sequence $\{\mathbf{w}^\nu\}$, with

$$\mathbf{w}^\nu = \frac{\lambda^\nu}{\|\lambda^\nu\|}, \text{ for all } \nu.$$

It is clear that $\mathbf{w}^\nu$ is bounded and hence without loss of generality we can conclude that $\mathbf{w}^\lambda \to \bar{\mathbf{w}}$ and $\|\mathbf{w}\| = 1$. Now observe that item (i) can be written as

$$\|\nabla f(\mathbf{x}^\nu) + \nabla g(\mathbf{x}^\nu)^T \lambda^\nu\| \leq \epsilon_\nu, \tag{8}$$

where $\nabla g(\mathbf{x})$ denotes the Jacobian matrix at the point $\mathbf{x}$ of the vector function $g : \mathbf{R}^n \to \mathbf{R}^m$, given as $g(\mathbf{x}) = (g_1(\mathbf{x}), \dots, g_m(\mathbf{x}))$.

Now by dividing both sides of eqn. 8 by $\|\lambda^\nu\|$ we have

$$\left\| \frac{1}{\|\lambda^\nu\|}\nabla f(\mathbf{x}^\nu) + \nabla g(\mathbf{x}^\nu)^T \frac{\lambda^\nu}{\|\lambda^\nu\|} \right\| \leq \frac{1}{\|\lambda^\nu\|}\epsilon_\nu$$

That is,

$$\left\| \frac{1}{\|\lambda^\nu\|}\nabla f(\mathbf{x}^\nu) + \nabla g(\mathbf{x}^\nu)^T \mathbf{w}^\nu \right\| \leq \frac{1}{\|\lambda^\nu\|}\epsilon_\nu. \tag{9}$$

Since $f$ is a smooth function as $\mathbf{x}^\nu \to \bar{\mathbf{x}}$ we have $\nabla f(\mathbf{x}^\nu) \to \nabla f(\bar{\mathbf{x}})$ and thus the sequence $\{\nabla f(\mathbf{x}^\nu)\}$ is bounded and further as $\epsilon_\nu \to 0$, the sequence $\{\epsilon_\nu\}$ is bounded. This shows that

$$\frac{1}{\|\lambda^\nu\|}\nabla f(\mathbf{x}^\nu) \to 0 \text{ as } \nu \to \infty$$

and

$$\frac{1}{\|\lambda^\nu\|}\epsilon_\nu \to 0 \text{ as } \nu \to \infty$$

Thus, passing to the limit in eqn. 9 as $\nu \to 0$, we have $\|\nabla g(\bar{\mathbf{x}})^T \bar{\mathbf{w}}\| \leq 0$ (note that since $g$ is smooth $\nabla g(\mathbf{x}^\nu) \to \nabla g(\bar{\mathbf{x}})$). That is, $\sum_{i=1}^m \bar{w}_i \nabla g_i(\bar{\mathbf{x}}) = 0$, where $\bar{\mathbf{w}} = (\bar{w}_1, \bar{w}_2, \dots, \bar{w}_m)$. Since $\|\mathbf{w}\| = 1$, it is clear that MFCQ is violated at $\mathbf{x} = \bar{\mathbf{x}}$. This is a contradiction. Hence, the sequence $\{\lambda^\nu\}$ is indeed bounded. Thus, we can assume without loss of generality that $\lambda^\nu \to \bar{\lambda} \in \mathbf{R}^m_+$ (since $\mathbf{R}^m_+$ is a closed set). Hence as $\nu \to \infty$ from items (1) and (2), we have

$$\|\nabla f(\bar{\mathbf{x}}) + \sum_{i=1}^m \bar{\lambda}_i \nabla g_i(\bar{\mathbf{x}})\| = 0,$$

and $\lambda_i g_i(\bar{\mathbf{x}}) = 0$, for $i = 1, 2, \dots, m$. Thus, we have

1. $\nabla f(\bar{\mathbf{x}}) + \sum_{i=1}^m \bar{\lambda}_i \nabla g_i(\bar{\mathbf{x}}) = 0$ and
2. $\lambda_i g_i(\bar{\mathbf{x}}) = 0$, for $i = 1, 2, \dots, m$.

Hence, $\bar{\mathbf{x}}$ is a KKT point.

**Remark** It is clear from the above theorem that if $f$ and each $g_i$, $i = 1, 2, \dots, m$ are differentiable convex functions, then $\bar{\mathbf{x}}$ as in the above theorem is a solution of the problem. Further, an important question is whether the sequence $\{\mathbf{x}^\nu\}$ will converge at all. Of course, if the set

$$C = \{x : g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \dots, m\}$$

is compact then $\{\mathbf{x}^\nu\}$ will have a subsequence which will converge and that would be enough for our purposes. Further, in many simple situations $C$ is actually compact. Consider for example,

$$C = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1, x + y \leq 1\}.$$

It is simple to observe that $C$ is compact.

### 4.2 Modified $\epsilon$- KKT Points: Nonsmooth Case

We will now consider the problem (P) with locally Lipschitz data. This means that both the objective $f$ and the constraints $g_i$ are locally Lipschitz. We will begin by introducing the notion of a modified $\epsilon$-KKT point.

**Definition 2** *A point $\bar{\mathbf{x}}$ which is feasible for (P) is said to be a modified $\epsilon$-KKT point for a given $\epsilon > 0$ if there exists $\hat{\mathbf{x}} \in \mathbf{R}^n$ such that $\|\bar{\mathbf{x}} - \hat{\mathbf{x}}\| \leq \sqrt{\epsilon}$ and there exists $\boldsymbol{\zeta} \in \partial^o f(\hat{\mathbf{x}})$ and $\boldsymbol{\psi}_i \in \partial^o g_i(\hat{\mathbf{x}})$ and scalars $\lambda_i \geq 0$ for $i = 1, \ldots, m$ such that*

1. $\|\boldsymbol{\zeta} + \sum_{i=1}^m \lambda_i \boldsymbol{\psi}_i\| \leq \sqrt{\epsilon}$, *and*

2. $\sum_{i=1}^m \lambda_i g_i(\bar{\mathbf{x}}) \geq -\epsilon$.

Interestingly, there is no restriction for $\hat{\mathbf{x}}$ to be feasible. Although the first condition is defined for $\hat{\mathbf{x}}$, the second condition must be true for the point $\bar{\mathbf{x}}$.

The above definition is motivated from the famous Ekeland's variational principle (EVP) which we now state below.

**Theorem 2** *(Ekeland's Variational Principle) Let $f : \mathbf{R}^n \to \mathbf{R}$ be a lower-semicontinuous function which is bounded below. Let $\epsilon > 0$ be given and let $\bar{\mathbf{x}} \in \mathbf{R}^n$ is such a point for which we have*

$$f(\bar{\mathbf{x}}) \leq \inf_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{x}) + \epsilon.$$

*Then for any $\gamma > 0$ there exists $\hat{\mathbf{x}} \in \mathbf{R}^n$ such that*

1. $\|\bar{\mathbf{x}} - \hat{\mathbf{x}}\| \leq \gamma$,

2. $f(\hat{\mathbf{x}}) \leq \inf_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{x}) + \epsilon$, *and*

3. $\hat{\mathbf{x}}$ *is the solution of the problem*

$$\min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{x}) + \frac{\epsilon}{\lambda} \|\mathbf{x} - \hat{\mathbf{x}}\|$$

The natural question to ask is whether the modified $\epsilon$-KKT point arises in a natural way. We show that at least in the case when (P) is a convex problem, it is indeed the case. We show this fact through the following theorem.

**Theorem 3** *Let us consider the problem (P) where $f$ and each $g_i$, $i = 1, , \ldots, m$ is a convex function. Let $\bar{\mathbf{x}}$ be a feasible point which is an $\epsilon$-minimum of (P). That is,*

$$f(\mathbf{x}) \leq \inf_{\mathbf{x} \in C} f(\mathbf{x}) + \epsilon.$$

*Assume further that the Slater's constraint qualification holds, that is, there exists a vector $\mathbf{x}^* \in \mathbf{R}^n$ such that $g_i(\mathbf{x}^*) < 0$, for all $i = 1, \ldots, m$. Then $\bar{\mathbf{x}}$ is a modified $\epsilon$-KKT point.*

**Proof** Since $\bar{\mathbf{x}}$ is an $\epsilon$-minimum of the convex problem it is clear that there is no $\mathbf{x} \in \mathbf{R}^n$ which satisfies the system

$$\begin{aligned} f(\mathbf{x}) - f(\bar{\mathbf{x}}) + \epsilon &< 0, \\ g_i(\mathbf{x}) &< 0, \quad i = 1, \ldots, m. \end{aligned}$$

Now using standard separation arguments (or the Gordon's theorem of the alternative) we conclude that there exists a vector $\mathbf{0}$ $(= (\lambda_0, \boldsymbol{\lambda}) \in \mathbf{R}_+ \times \mathbf{R}^m_+$ such that for all $\mathbf{x} \in \mathbf{R}^n$

$$\lambda_0(f(\mathbf{x}) - f(\bar{\mathbf{x}})) + \lambda_0 \epsilon + \sum_{i=1}^{n} \lambda_i g_i(\mathbf{x}) \geq 0. \tag{10}$$

We claim that $\lambda_0 = 0$ and hence from eqn. 10 we have

$$\sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in \mathbf{R}^n.$$

However, it is clear that $\sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}^*) < 0$. Hence, $\lambda_0 > 0$ and one can take $\lambda_0 = 1$ without loss of generality. This shows that

$$(f(\mathbf{x}) - f(\bar{\mathbf{x}})) + \epsilon + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in \mathbf{R}^n. \tag{11}$$

Now putting $\mathbf{x} = \bar{\mathbf{x}}$, we have

$$\sum_{i=1}^{m} \lambda_i g_i(\bar{\mathbf{x}}) \geq -\epsilon.$$

This establishes item 2 in the definition of a modified $\epsilon$-KKT point. Now setting,

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}),$$

we have from eqn. 11

$$L(\mathbf{x}, \boldsymbol{\lambda}) \geq L(\bar{\mathbf{x}}, \boldsymbol{\lambda}) - \epsilon, \quad \forall \mathbf{x} \in \mathbf{R}^n. \tag{12}$$

Thus, $\bar{\mathbf{x}}$ is the $\epsilon$-minimum of $L(\cdot, \boldsymbol{\lambda})$ over $\mathbf{R}^n$. Now applying the Ekeland's variational principle we have by setting $\gamma = \sqrt{\epsilon}$, that there exists $\hat{\mathbf{x}} \in \mathbf{R}^n$ such that $\|\bar{\mathbf{x}} - \hat{\mathbf{x}}\| \sqrt{\epsilon}$ and $\hat{\mathbf{x}}$ solves the convex problem

$$\min_{\mathbf{x} \in \mathbf{R}^n} \; L(\mathbf{x}, \boldsymbol{\lambda}) + \sqrt{\epsilon} \|\mathbf{x} - \hat{\mathbf{x}}\|.$$

Thus from standard rules of convex analysis we have

$$\mathbf{0} \in \partial L(\hat{\mathbf{x}}, \lambda) + \sqrt{\epsilon} \mathbf{B}_{\mathbf{R}^n},$$

where $\mathbf{B}_{\mathbf{R}^n}$ denotes the unit ball in $\mathbf{R}^n$. Hence, using again usual rules of convex analysis, we have

$$\mathbf{0} \in \partial f(\hat{\mathbf{x}}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\hat{\mathbf{x}}) + \sqrt{\epsilon}\mathbf{B}.$$

Thus there exists $\boldsymbol{\zeta} \in \partial f(\hat{\mathbf{x}})$ and $\boldsymbol{\psi}_i \in \partial g_i(\hat{\mathbf{x}})$ and $\mathbf{b} \in \mathbf{B}_{\mathbf{R}^n}$ such that

$$\mathbf{0} = \boldsymbol{\zeta} + \sum_{i=1}^{m} \lambda_i \boldsymbol{\psi}_i + \sqrt{\epsilon}\mathbf{b}.$$

Hence, $\|\boldsymbol{\zeta} + \sum_{i=1}^{m} \lambda_i \boldsymbol{\psi}_i\| \leq \sqrt{\epsilon}$. This establishes the result.

If we consider $\hat{\mathbf{x}} = \bar{\mathbf{x}}$ in the above definition, any positive $\epsilon$ value will satisfy $\|\bar{\mathbf{x}} - \hat{\mathbf{x}}\| \leq \sqrt{\epsilon}$. Since $\boldsymbol{\zeta}$ and $\boldsymbol{\psi}_i$ are now to be chosen from sub-gradients of $f$ and $g_i$ at $\bar{\mathbf{x}}$, respectively, we can use the first condition to set $\epsilon$ such that it is the maximum of two quantities: (i) the minimum squared value of $\|\boldsymbol{\zeta} + \sum_{i=1}^{m} \lambda_i \boldsymbol{\psi}_i\|$ and (ii) $\sum_{i=1}^{m} \lambda_i g_i(\bar{\mathbf{x}})$, over $\boldsymbol{\zeta} \in \partial^o f(\bar{\mathbf{x}})$, $\boldsymbol{\psi}_i \in \partial^o g_i(\bar{\mathbf{x}})$ and $\lambda_i \geq 0$ for $i = 1, \ldots, m$.

Before stating the next result let us mention the non-smooth version of MFCQ that we need in this sequel. we shall call this this the basic constraint qualification. The problem (P) satisfies the basic constraint qualification at $\bar{\mathbf{x}}$ if there exists no $\boldsymbol{\lambda} \in \mathbf{R}^m_+$ with $\boldsymbol{\lambda} \neq 0$ and $\lambda_i \geq 0$, for all $i \in I(\mathbf{x})$ and $\lambda_i = 0$ for $i \notin I(\bar{\mathbf{x}})$ such that

$$\mathbf{0} \in \sum_{i=1}^{m} \lambda_i \partial^o g_i(\bar{\mathbf{x}}).$$

**Theorem 4** *Let us consider the problem (P) with locally Lipschitz objective function and constraints. Let $\{\mathbf{x}^\nu\}$ be a sequence of vectors feasible to (P) and let $\mathbf{x}^\nu \to \bar{\mathbf{x}}$ as $\nu \to \infty$. Consider $\{\epsilon_\nu\}$ to be a sequence of positive real numbers such that $\epsilon_\nu \downarrow 0$ as $\nu \to \infty$. Further assume that for each $\nu$, $\mathbf{x}^\nu$ is a modified $\epsilon_\nu$-KKT point of (P). Let the Basic Constraint qualification hold at $\bar{\mathbf{x}}$. Then $\bar{\mathbf{x}}$ is a KKT point of (P).*

**Proof** Since each $\mathbf{x}^\nu$ is a modified $\epsilon_\nu$-KKT point, for each $\nu$ there exists $\mathbf{y}^\nu$ with $\|\mathbf{x}^\nu - \mathbf{y}^\nu\| \leq \sqrt{\epsilon_\nu}$ and there exists $\boldsymbol{\zeta}^\nu \in \partial^o f(\mathbf{y}^\nu)$, $\boldsymbol{\psi}_i^\nu \in \partial^o g_i(\mathbf{y}^\nu)$, $i = 1, 2, \ldots m$ and scalars $\lambda_i^\nu \geq 0$, $i = 1, 2 \ldots m$ such that

1. $\|\boldsymbol{\zeta}^\nu + \sum_{i=1}^{m} \lambda_i^\nu \boldsymbol{\psi}_i^\nu\| \leq \sqrt{\epsilon_\nu}$
2. $\sum_{i=1}^{m} \lambda_i^\nu g_i(\mathbf{x}^\nu) \geq -\epsilon_\nu$

Let us first show that $\{\lambda^\nu\}$ is bounded. We assume on the contrary that $\{\lambda^\nu\}$ is unbounded. Thus, without loss of generality, let us assume that $\|\lambda^\nu\| \to \infty$ as $\nu \to \infty$. Now consider the sequence $w^\nu = \frac{\lambda^\nu}{\|\lambda^\nu\|}$. Then $\{w^\nu\}$ is a bounded sequence and hence has a converget subsequence. Thus, without loss of generality we can assume that $w^\nu \to w$ and $\|w\| = 1$. Now observe the following:

$$\|\mathbf{y}^\nu - \bar{\mathbf{x}}\| \leq \|\mathbf{y}^\nu - \mathbf{x}^\nu\| + \|\mathbf{x}^\nu - \bar{\mathbf{x}}\|$$

Hence,

$$\|\mathbf{y}^\nu - \bar{\mathbf{x}}\| \leq \sqrt{\epsilon_\nu} + \|\mathbf{x}^\nu - \bar{\mathbf{x}}\|$$

Now as $\nu \to \infty$, $\epsilon_\nu \downarrow 0$ and $\mathbf{x}^\nu \to \bar{\mathbf{x}}$. This shows that $\mathbf{y}^\nu \to \bar{\mathbf{x}}$. Since, the Clarke subdifferential is locally bounded, the sequence $\{\boldsymbol{\zeta}\}$ and $\{\boldsymbol{\psi}_i^\nu\}$, for $i = 1, 2, \ldots m$ are bounded. Thus, without loss of generality we can conclude that $\boldsymbol{\psi}_i^\nu \to \bar{\boldsymbol{\psi}}_i$ for all $i = 1, 2, \ldots m$. Further, as $\partial^o g_i(\bar{\mathbf{x}})$. From 1 we have:

$$\frac{1}{\|\lambda^\nu\|}\|\boldsymbol{\zeta}^\nu + \sum_{i=1}^m \lambda_i^\nu \boldsymbol{\psi}_i^\nu\| \le \sqrt{\epsilon_\nu}$$

Thus,

$$\|\frac{1}{\|\lambda^\nu\|}\boldsymbol{\zeta}^\nu + \sum_{i=1}^m w_i^\nu \boldsymbol{\psi}_i^\nu\| \le \frac{1}{\|\lambda^\nu\|}\sqrt{\epsilon_\nu}$$

where $w^\nu = (w_1^\nu, w_2^\nu, \ldots w_m^\nu)$, and $w^\nu = \frac{\lambda^\nu}{\|\lambda^\nu\|}$. Now as $\epsilon_\nu \downarrow 0$, $\sqrt{\epsilon_\nu} \to 0$ and hence, $\{\sqrt{\epsilon_\nu}\}$ is a bounded sequence and

$$\frac{1}{\|\lambda^\nu\|}\sqrt{\epsilon_\nu} \to 0 \ as \ \nu \to \infty$$

Further as $\{\boldsymbol{\zeta}^\nu\}$ is a bounded sequence, we have $\frac{1}{\|\lambda^\nu\|}\boldsymbol{\zeta}^\nu \to 0$ as $\nu \to \infty$. Hence from 4.2 we have

$$\|\sum_{i=1}^m w_i \bar{\boldsymbol{\psi}}_i\| \le 0$$

i.e. $\sum_{i=1}^m w_i \bar{\boldsymbol{\psi}}_i = 0$ where $w = (w_1, w_2, \ldots w_m)$ and $w^\nu \to w$. As $\|w\| = 1$, it shows the Basic constraint qualification is violated. Hence, $\{\lambda^\nu\}$ is a bounded sequence and thus we can assume that $\lambda^\nu \to \bar{\lambda}$, $\bar{\lambda} \in \mathbf{R}_+^m$. Further as $\{\boldsymbol{\zeta}^\nu\}$ is bounded we can assume that $\boldsymbol{\zeta}^\nu \to \bar{\boldsymbol{\zeta}}$ and since $\partial^o f$ is graph closed, $\bar{\boldsymbol{\zeta}} \in \partial^o f(\bar{\mathbf{x}})$. Hence, from 1 we can have $\|\bar{\boldsymbol{\zeta}} + \sum_{i=1}^m \bar{\lambda}_i \bar{\boldsymbol{\psi}}_i\| \le 0$, where $\bar{\lambda} = (\bar{\lambda_1}, \bar{\lambda_2}, \ldots, \bar{\lambda_m})$. Thus, $\bar{\boldsymbol{\zeta}} + \sum_{i=1}^m \bar{\lambda}_i \bar{\boldsymbol{\psi}}_i = 0$. This shows that $\mathbf{0} \in \partial^o f(\bar{\mathbf{x}}) + \sum_{i=1}^m \bar{\lambda}_i \partial^o g_i(\bar{x})$. From 2 we have $\sum_{i=1}^m \bar{\lambda}_i g_i(\bar{\mathbf{x}}) \ge 0$. Now since $\mathbf{x}^\nu$ is feasible we have $g_i(\mathbf{x}^\nu) \le 0$, for all $i = 1, 2, \ldots m$. Hence, $g_i(\bar{x}) \le 0$ for all $i = 1, 2, \ldots m$. Hence, $\bar{x}$ is feasible to (P). Since $\bar{\lambda}_i \ge 0$, we have $\sum_{i=1}^m \bar{\lambda}_i g_i \bar{\mathbf{x}} \le 0$. This shows that,

$$\sum_{i=1}^m \bar{\lambda}_i g_i(\bar{\mathbf{x}}) = 0$$

Hence $\bar{x}$ is a KKT point of (P).

**Theorem 5** *Let us consider the convex problem (P). Let the Slater's constraint qualification holds, that is, there exists* $\mathbf{x}^* \in \mathbf{R}^n$ *such that* $g_i(\mathbf{x}^*) < 0$ *for all* $i = 1, \ldots, m$. *Let* $\{\mathbf{x}^\nu\}$ *be a sequence of* $\epsilon_\nu$*-minimum points and let* $\mathbf{x}^\nu \to \bar{\mathbf{x}}$ *as* $\epsilon_\nu \downarrow 0$. *Then,* $\bar{\mathbf{x}}$ *is a minimum point of (P).*

**Proof** Since $\mathbf{x}^\nu$ is an $\epsilon_\nu$-minimum point by applying Theorem 3 we have $\mathbf{x}^\nu$ is modified $\epsilon_\nu$-KKT point for (P) for each $\nu$. Now applying Theorem 4 we have $\bar{\mathbf{x}}$ is a KKT point for (P) and hence $\bar{\mathbf{x}}$ is a solution of (P). Note that that Slater's Constraint Qualification implies that Basic Constraint qualification is the case of a convex optimization problem.

### 4.3 Andreani et al.'s Definition

Andreani [8] defined an approximate KKT point as follows:

**Definition 1** *A feasible point* $\mathbf{x}^*$ *satisfies approximate KKT condition, if, and only if, there exist* *sequences of feasible solutions* $\{\mathbf{x}^k\} \subset R^n$, $\{\mathbf{u}\} \subset R_+^m$ *and* $\{\epsilon_k\} \subset R_+$ *such that* $\mathbf{x}^k \to \mathbf{x}^*$, $\epsilon_k \to 0$ *and for all* $k \in I\!N$,

$$\|\nabla f(\mathbf{x}^k) + \sum_{j=1}^{m} u^k \nabla g(\mathbf{x}^k)\| \leq \epsilon_k, \tag{13}$$

$$u_j^k = 0, \textit{for all } j \textit{ such that } g_j(\mathbf{x}^k) < -\epsilon_k. \tag{14}$$

These conditions differ from KKT conditions in the previous section in the sense that they relax the multiplier $u_j$ to be nonzero for some feasible points, lying in an $\epsilon_k$-proximity to the $j$-th constraint boundary.

Lets denote the level of *complimentary slackness relaxation* by parameter $s_k$ such that we require $u_j^k = 0$ for all $j$ satisfying $g_j(\mathbf{x}^k) < -s_k$. Note that, for a typical problem having lower and upper limit constraints on the decision variables, if we relax the complimentary slackness completely ($s_k = \infty$) we are bound to get a zero KKT-Error for any feasible point. This happens because the gradients of limit constraints by themselves, span the entire decision space and hence can satisfy the equilibrium equation for any point.

Thus, for a given $\mathbf{x}^k$, a larger value of $s_k$ allows a larger set of constraints to have non-zero multipliers and participate in the equilibrium equation, thereby resulting in a smaller value of KKT-Error. On the other hand, too small a value of $s_k$ will only allow nearly active constraints to participate, resulting in high KKT-Error values for most of the feasible space, as we have seen in section 2.1. Thus, for every solution $\mathbf{x}^k$, there will exist an $\epsilon_k$ which will include adequate constraints into the equilibrium equation to make the KKT-Error smaller than $\epsilon_k$.

Interestingly, the above definition suggests that if an algorithm produces a sequence of points $\mathbf{x}^k$ and if it is possible to find corresponding $\epsilon_k$ values exhibiting a reducing sequence to zero, then the limit of the sequence $\{\mathbf{x}^k\}$ is an approximate KKT point, termed as AKKT point.

The application of the above definition to an evolutionary algorithm is that the sequence of points $\{\mathbf{x}^k\}$ can be considered as the set of generation-wise ($k$) best solutions. Thereafter, if we can find a reducing sequence of $\epsilon_k$ for the EA iterates satisfying [8], and if the $\epsilon_k$ for the final generation is close to zero, the corresponding final generation best solution is an approximate KKT point. Although the above connection of the KKT theory with the evolutionary algorithm is interesting, it is not clear how to obtain such a reducing sequence of $\epsilon_k$ values.

The above relaxation of complimentary slackness condition takes the discontinuity of $u_j$ away from the constraint boundary, but at $g_j(\mathbf{x}^k) = -\epsilon_k$, $u_j$ is still discontinuous.

## 5 Optimization Toolbox & Approximate KKT Error

KKT conditions are used as stopping criterion in a number of commerically available softwares and toolboxes[7, 5]. Knitro, a widely used package integrates two powerful and complementary algorithmic approaches for nonlinear optimization: the interior-point approach and the active-set approach, employing conjugate gradient iteration in the step computation. It uses KKT first order conditions to identify a locally optimal solution, and therefore as a terminating criterion [7].

For the problem 1, Knitro defines the feasibility error(`FeasErr`) at a point $\mathbf{x}^k$ to be the maximum violation of the contraints, ie:

$$FeasErr = \max_{i=1...m}(0, g_i(\mathbf{x}^k))$$

and the optimality error(`OptErr`) as:

$$OptErr = \max_{i=1...m}(\|\nabla f(\mathbf{x}^k) + \sum_{j=1}^{m} u_j \nabla g_j(\mathbf{x}^k)\|_\infty, -u_i g_i(\mathbf{x}^k))$$

with $u_j \geq 0$ enforced explicitly throughout the optimization. Furthermore, `FeasErr` and `OptErr` are scaled using

$$\tau_1 = \max_{i=1...m}(1, g_i(\mathbf{x}^0))$$

$$\tau_2 = \max(1, \|\nabla f(\mathbf{x}^k)\|_\infty)$$

where $\mathbf{x}^0$ represents the initial point.

Based on constants defined by the user options `feastol, opttol`, Knitro declares a locally optimal solution iff:

$$FeasErr \leq \tau_1 * feastol$$

$$OptErr \leq \tau_2 * opttol$$

However, it is interesting to note that at no point in time in the algorithm, to compute `OptErr` are we computing $u_j$. These Lagrange multipliers are obtained as a side computation of the approximate solution to the quadratic model which is performed in every iteration of an Sequential Quadratic Programming method with trust-region[1].

Similarly MATLAB employs first-order optimality as a measure of the closeness of $\mathbf{x}$ to the optima, with the paramter `TolFun` relating to first order optimality measure. For more information on first-order optimality conditions, please see [6].

# 6 Simulation Results

## 6.1 A Schematic Approach For Computing KKT-Proximity Measure

In this section, we suggest one such $\epsilon_k$ reduction scheme which gives us a sequence of modified $\epsilon$-KKT points. It must be noted that for computational purposes in the scheme, we have taken $\hat{\mathbf{x}}$ given in the definition of modified $\epsilon$-KKT points as equal to $\mathbf{x}$.

To find that critical $\epsilon_k$ for the modified $\epsilon$-KKT point, we solve the following optimization problem with $(\epsilon_k, \mathbf{u})$ as the variable vector for the best feasible solution $\mathbf{x}^k$ at every generation:

$$\begin{aligned} \text{Minimize} \quad & \epsilon_k \\ \text{Subject to} \quad & \|\nabla f(\mathbf{x}^k) + \textstyle\sum_{j=1}^m u_j \nabla g_j(\mathbf{x}^k)\|^2 \leq \epsilon_k, \\ & \textstyle\sum_{j=1}^m u_j g_j(\mathbf{x}^k) \geq -\epsilon_k, \\ & u_j \geq 0, \quad \forall j. \end{aligned} \tag{15}$$

We present the optimum $\epsilon_k$ value as the KKT-Proximity measure for the iterate $\mathbf{x}^k$.

### 6.2 Results

We investigate the results of the above-mentioned $\epsilon_k$-reduction scheme with modified $\epsilon$-KKT definition of an approximate KKT point, on a demonstration problem.

We consider a simple two-variable, two-constraint problem to illustrate the working of our scheme:

$$\begin{aligned} \text{Minimize} \quad f(x_1, x_2) &= x_1^2 + x_2^2, \\ \text{s.t.} \quad g_1(x_1, x_2) &= 3x_1 - x_2 + 1 \leq 0, \\ g_2(x_1, x_2) &= x_1^2 + (x_2 - 2)^2 - 1 \leq 0. \end{aligned}$$

Figure 3 shows the feasible space and the optimum point ($\mathbf{x}^* = (0, 1)^T$). We consider a series

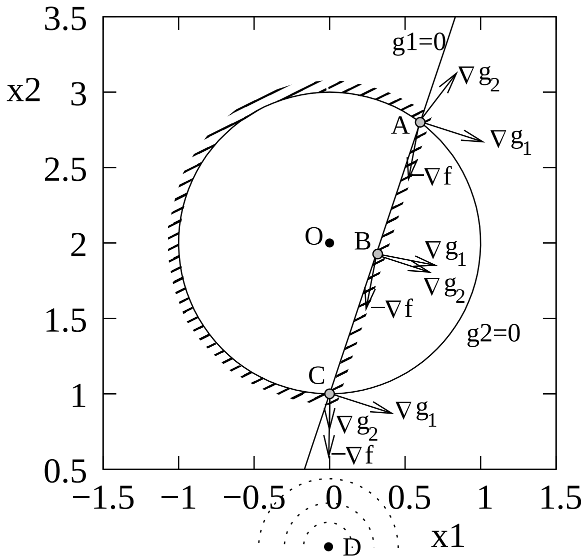



Figure 3: Feasible search space of the example problem.

of iterates (point number 0 at A to point number 49 at C) along the linear constraint boundary from A to C and investigate the behavior of KKT error estimate using above-mentioned scheme and compare it with the complementary slackness scheme, discussed in subsection 3. It is clear from the figure that point A will have a large KKT-proximity value, as no linear combination of $\nabla g_1$ and $\nabla g_2$ vectors will construct $-\nabla f$ at this point. However, as points towards C and inside the circle is considered, the second constraint is inactive and it will have no role to play in the KKT conditions. Thus, for points inside the circle and on the first constraint, only the first constraint participates in the KKT-proximity calculation for the complementary slackness scheme. It is clear that on none of these points, $\nabla g_1 = -\nabla f$ in order to make a zero error. In fact, the KKT-proximity measure reduces from near point A to near point C, as shown in Figure 4. At point C, the second constraint is active again and $\nabla g_2$ is equal to $-\nabla f$ at C. Thus,

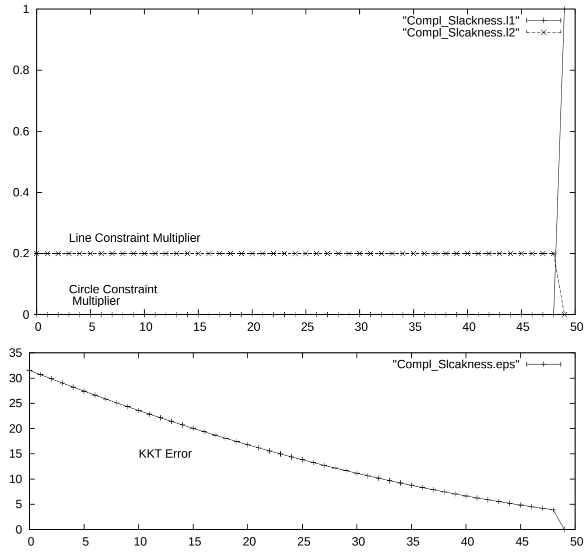



Figure 4: Lagrange multipliers and KKT error using the complementary slackness scheme.

the KKT-proximity measure will be zero. As it is clear from the figure, in the neighborhood of C, although KKT-proximity measure reduces as the point gets closer to C along the line AC, there is a discontinuity of the measure at C. Lagrange multipliers for $g_1$ and $g_2$ are also shown in Figure 4. Since the second constraint is inactive for points along AC (except A and C), $u_2$ is zero and at C it is a positive value ($u_2 = 1$). Interestingly, $u_1$ is constant ($= 0.2$) throughout, except at C, at which it is zero. The KKT-proximity measure varies as $40x_1^2 + 24x_1 + 3.6$ in the range $0 < x_1 < 0.6$ and at a point near C ($x_1 = 0$) the measure is near 3.6. Then exactly at C, the measure is zero, making a jump in the error value from near 3.6 to zero, as shown in the figure.

Now, we illustrate our proposed scheme for computing the KKT-proximity measure. Figure 5 shows that proximity measure reduces as the solutions progress towards the KKT point C (also the minimum point, in this case). Our proposed approach seems to maintain a continuity in the

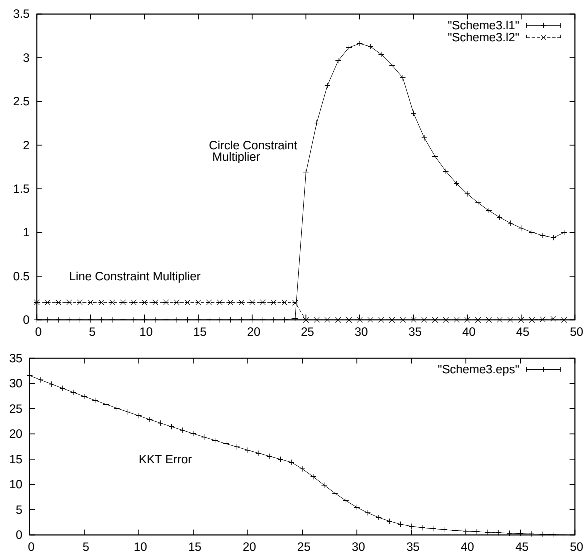



Figure 5: Lagrange multipliers and KKT error using scheme 3.

KKT-proximity measure as it reduces to zero. Due to this property, this modified KKT-proximity measure can be used in the termination criterion of an optimization algorithm. Corresponding Lagrange multipliers $u_1$ and $u_2$ are also shown in Figure 5. Interestingly, $u_2$ is zero from A till an intermediate point B. As shown in Figure 3, the gradient of constraint $g_1$ is more directed towards $-\nabla f$ and contributes in minimizing the KKT error. At around B, the role of $g_1$ and $g_2$ get interchanged and $g_2$ is more directed towards $-\nabla f$, thereby making its Lagrange multiplier nonzero. Since the scheme allows larger values of $u_2$, it grows to the extent needed to reduce the KKT-proximity measure. Note that unless $\nabla g_2$ is equal to $-\nabla f$, the KKT-proximity measure can never be exactly zero, but due to the flexibility in choosing a large enough $u_2$, the KKT-proximity measure smoothly reduces to zero.

### 6.3 KNITRO

As discussed above, the Knitro software package computes the Lagrange multipliers in every iteration of its optimization routines, and computes a KKT-Proximity measure of its own, which it uses as terminating condition. Here, we conduct a comparative study on the problem given below.

$$
\begin{aligned}
\text{Minimize:} \quad f(\mathbf{x}) &= 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \\
\text{Subject To:} \quad g_1(\mathbf{x}) &= 1 - x_1 x_2 \leq 0 \\
g_2(\mathbf{x}) &= -x_1 - x_2^2 \leq 0 \\
g_3(\mathbf{x}) &= x_1 - 0.5 \leq 0
\end{aligned}
$$

Table 1 contains the KKT-proximity measure values obtained using our scheme and those given by Knitro for the same set of intermediate points. It is observed that while the KKT-proximity measure values show the same trends, their absolute magnitudes differ significantly, thereby confirming the novelty of our scheme.

### 6.4 Numerical Results: Smooth Problems

The procedure proposed in sections 6.1 is tested on a variety of problems with inequality constraints.

We take the sequence of best individual of the population, for different iterates of a real-coded genetic algorithm, adapted for handling constraints using a penalty-parameter-less strategy [9]. For the GA runs, we first delete contiguously duplicate solutions and then solutions which are infeasible. Thereafter, to each remaining solution, we apply the proposed scheme to compute the discussed KKT-proximity measure. For problems in which the GA run does not converge to the optimum, the reported optimal solution from [13] is manually appended at the end of the GA sequence of solutions $\{\mathbf{x}^k\}$. This is done to mainly demonstrate the accuracy of the computation scheme, in checking whether the proximity measure goes to zero for the optimum point or not.

Table 1: Comparison of KNitro and Scheme KKT-proximity measure values (For no.1-4 fmincon did not seem to converge after minutes of running.)

| No. | $\mathbf{x}$ | | | KKT Error | |
|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | Feasible | Scheme 1 | KNitro |
| 1 | 0.350000 | 10000.0 | Yes | - | - |
| 2 | 0.392786 | 5025.00 | Yes | - | 37.750 |
| 3 | 0.487528 | 2525.060 | Yes | - | 12.920 |
| 4 | 0.494462 | 1268.840 | Yes | - | 3.6280 |
| 5 | 0.498565 | 637.59400 | Yes | 3.8655e+07 | 0.936300 |
| 6 | 0.499620 | 320.39000 | Yes | 1.1828e+07 | 0.925500 |
| 7 | 0.499902 | 160.99500 | Yes | 2.4617e+06 | 0.961900 |
| 8 | 0.499975 | 80.898700 | Yes | 6.2062e+05 | 0.961900 |
| 9 | 0.499994 | 40.648500 | Yes | 1.5623e+05 | 0.986600 |
| 10 | 0.499998 | 20.419800 | Yes | 3.9212e+04 | 0.986600 |
| 11 | 0.499999 | 10.248700 | Yes | 9.7868e+03 | 0.517800 |
| 12 | 0.499997 | 5.125720 | Yes | 2.4183e+03 | 0.129300 |
| 13 | 0.499984 | 2.527150 | Yes | 232.0659 | 0.032160 |
| 14 | 0.483281 | 2.086940 | Yes | 38.7791 | 0.019330 |
| 15 | 0.499916 | 2.018170 | Yes | 6.4107 | 0.001846 |
| 16 | 0.500000 | 2.000080 | Yes | 0.0280 | 0.0000393 |
| 17 | 0.500000 | 2.000000 | Yes | 1.9996e-23 | 0.00 |

### 6.4.1 Problem *g01*

The following is Problem g01 from [13] containing 35 constraints and 13 variables.

Minimize:

$$f(\mathbf{x}) = 5\sum_{i=1}^{4} x_i - 5\sum_{i=1}^{4} {x_i}^2 - \sum_{i=5}^{13} x_i$$

subject to:

$$
\begin{aligned}
g_1(\mathbf{x}) &= 2x_1 + 2x_2 + x_{10} + x_{11} - 10 \leq 0 \\
g_2(\mathbf{x}) &= 2x_1 + 2x_3 + x_{10} + x_{12} - 10 \leq 0 \\
g_3(\mathbf{x}) &= 2x_2 + 2x_3 + x_{11} + x_{12} - 10 \leq 0 \\
g_4(\mathbf{x}) &= -8x_1 + x_{10} \leq 0 \\
g_5(\mathbf{x}) &= -8x_2 + x_{11} \leq 0 \\
g_6(\mathbf{x}) &= -8x_3 + x_{12} \leq 0 \\
g_7(\mathbf{x}) &= -2x_4 - x_5 + x_{10} \leq 0 \\
g_8(\mathbf{x}) &= -2x_6 - x_7 + x_{11} \leq 0 \\
g_9(\mathbf{x}) &= -2x_8 - x_9 + x_{12} \leq 0 \\
g_{8+2i}(\mathbf{x}) &= -x_i \leq 0 \quad (\mathrm{i} = 1, \ldots, 13) \\
g_{8+2i+1}(\mathbf{x}) &= x_i - 1 \leq 0 \quad (\mathrm{i} = 1, \ldots, 9, 13) \\
g_{8+2i+1}(\mathbf{x}) &= x_i - 100 \leq 0 \quad (\mathrm{i} = 10, 11, 12)
\end{aligned}
$$

Outputs from 10 GA runs are taken and the median, best and worst KKT proximity measures are plotted in figure 6. In all the runs, the GA converges to the optimum point in this problem ($f^* = -15$). Despite some initial fluctuations in the KKT error, it finally reduced to zero, indicating that the final GA point is a KKT point. For this problem, we observe that six constraints $\{g_1,\ g_2,\ g_3,\ g_7,\ g_8,\ g_9\}$ including 10 upper-limit constraints $\{x_i \leq 1,\ \text{for}\ i = 1, \ldots, 9, 13\}$ are active at the optimum.

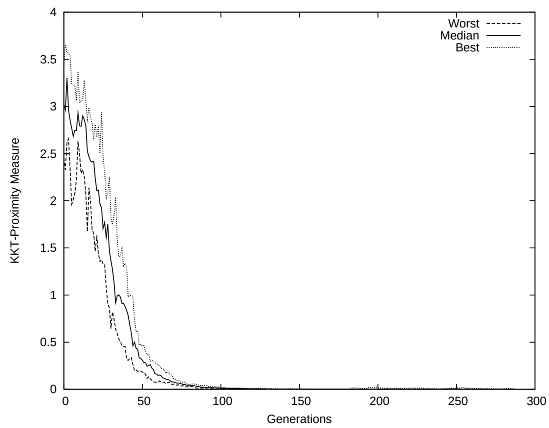



Figure 6: KKT-proximity measure for problem g01.

### 6.4.2 Problem *hs23*

The following problem is hs23 from [4]. It is a 2 variable, 9 constraint problem with quadratic objective function and smooth contraints.

Minimize:

$$
f(\mathbf{x}) = x_1^2 + x_2^2
$$

subject to:

$$\begin{aligned}
g_1(\mathbf{x}) &= 1 - x_1 - x_2 \leq 0 \\
g_2(\mathbf{x}) &= 1 - x_1^2 - x_2^2 \leq 0 \\
g_3(\mathbf{x}) &= 9 - 9x_1^2 - x_2^2 \leq 0 \\
g_4(\mathbf{x}) &= x_2 - x_1^2 \leq 0 \\
g_5(\mathbf{x}) &= x_1 - x_2^2 \leq 0 \\
g_{4+2i}(\mathbf{x}) &= -50 - x_i \leq 0 \quad (\mathrm{i} = 1, 2) \\
g_{4+2i+1}(\mathbf{x}) &= x_i - 50 \leq 0 \quad (\mathrm{i} = 1, 2)
\end{aligned}$$

Again, output from 10 GA runs are taken and the best, worst and median KKT proximity measure are plotted in figure 7. In all the runs, the GA converges to the optima at [1.0 1.0] and the KKT proximity measure converges to zero between the $30 - 70$ generations. At the optima $g_4$ and $g_5$ are active.

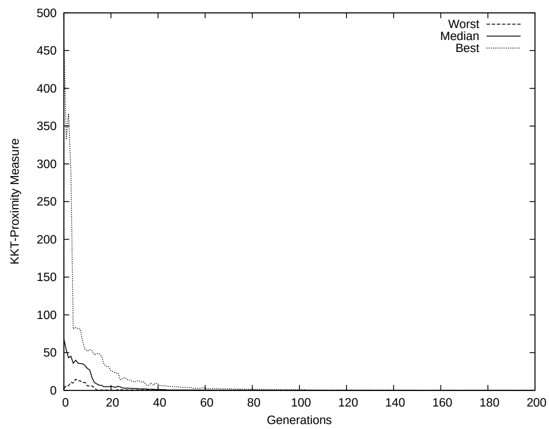



Figure 7: KKT-proximity measure for problem hs23.

#### 6.4.3 Problem *hs45*

The following is problem hs45 from [4]. It is a 5 variable, 10 constaint problem.

Minimize:

$$f(\mathbf{x}) = 2 - \frac{1}{120} x_1 x_2 x_3 x_4 x_5$$

subject to:

$$\begin{aligned}
g_{2i-1}(\mathbf{x}) &= -x_i \leq 0 \quad (\mathrm{i} = 1, \ldots, 5) \\
g_{2i}(\mathbf{x}) &= x_i - i \leq 0 \quad (\mathrm{i} = 1, \ldots, 5)
\end{aligned}$$

The results in figure 8 indicate that the best, median and worst KKT proximity measure converge to zero close to the $30^{th}$ generation, after large initial fluctuations. All the runs converge to the optima at [1.0 2.0 3.0 4.0 5.0] and $g_6,\ g_7,\ g_8,\ g_9,\ g_{10}$ are active. Since, its evident from the diagram that the worst and the best KKT proximity measure display a similar behaviour, from now on we will only consider a singular GA run and plot the KKT proximity measure.

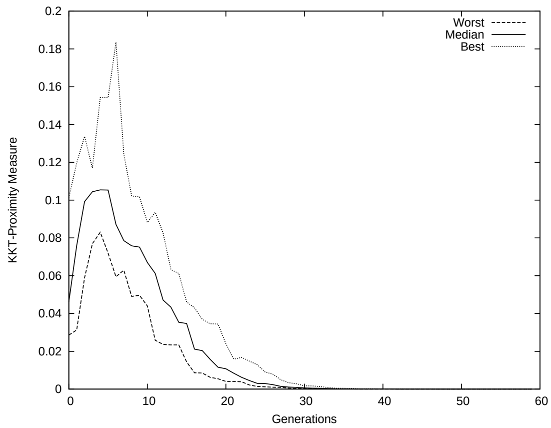



Figure 8: KKT-proximity measure for problem hs45.

#### 6.4.4 Problem *g02*

The following is problem g02 from [13]. It is a 20 variable, 2 constraint problem, besides variable bounds.

Minimize:

$$f(\mathbf{x}) = -\left|\frac{\sum_{i=1}^{20} cos^4(x_i) - 2\prod_{i=1}^{20} cos^2(x_i)}{\sqrt{\sum_{i=1}^{20} ix_i{}^2}}\right|$$

subject to:

$$\begin{aligned} g_1(\mathbf{x}) &= 0.75 - \prod_{i=1}^{20} x_i \leq 0 \\ g_2(\mathbf{x}) &= \sum_{i=1}^{20} x_i - 7.5 * 20 \leq 0 \\ g_{1+2i}(\mathbf{x}) &= -x_i \leq 0 \quad (\text{i} = 1, \ldots, 20) \\ g_{1+2i+1}(\mathbf{x}) &= x_i - 10 \leq 0 \quad (\text{i} = 1, \ldots, 20) \end{aligned}$$

For this problem, the GA converges to a point at an Euclidean distance of 2.3568 from the reported optimum [13]. At the manually appended optimum, the proximity measure is found to be zero (Figure 9), meaning that reported minimum ($f^* = -0.80362$) is a likely candidate for the optimum. The first constraint $g_1$ is found to be active with $u_1 = 0.0470$.

#### 6.4.5 Problem *g04*

The following is problem g04 from [13], containing 5 variables and 6 constraints besides variable bounds.

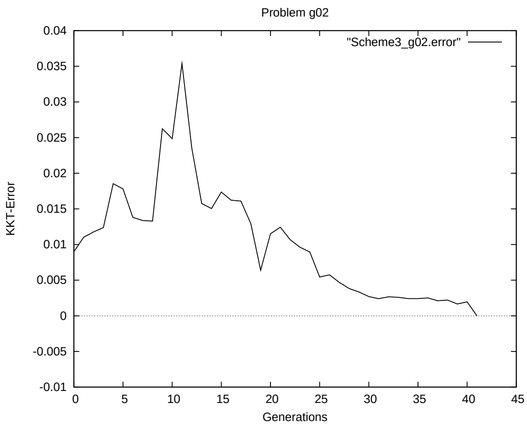



Figure 9: KKT-proximity measure for problem g02.

Minimize:

$$
\begin{aligned}
f(\mathbf{x}) &= 5.3578547{x_3}^2 + 0.8356891x_1x_5 + \\
&\quad 37.293239x_1 - 40792.141
\end{aligned}
$$

subject to:

$$
\begin{aligned}
g_1(\mathbf{x}) &= 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - \\
&\quad 0.0022053x_3x_5 - 92 \leq 0 \\
g_2(\mathbf{x}) &= -85.334407 - 0.0056858x_2x_5 - 0.0006262x_1x_4 + \\
&\quad 0.0022053x_3x_5 \leq 0 \\
g_3(\mathbf{x}) &= 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + \\
&\quad 0.0021813{x_3}^2 - 110 \leq 0 \\
g_4(\mathbf{x}) &= -80.51249 - 0.0071317x_2x_5 - 0.0029955x_1x_2 - \\
&\quad 0.0021813{x_3}^2 + 90 \leq 0 \\
g_5(\mathbf{x}) &= 9.300961 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + \\
&\quad 0.0019085x_3x_4 - 25 \leq 0 \\
g_6(\mathbf{x}) &= -9.300961 - 0.0047026x_3x_5 - 0.0012547x_1x_3 - \\
&\quad 0.0019085x_3x_4 + 20 \leq 0 \\
g_7(\mathbf{x}) &= 78 - x_1 \leq 0 \\
g_8(\mathbf{x}) &= x_1 - 102 \leq 0 \\
g_9(\mathbf{x}) &= 33 - x_2 \leq 0 \\
g_{10}(\mathbf{x}) &= x_2 - 45 \leq 0 \\
g_{5+2i}(\mathbf{x}) &= 27 - x_i \leq 0 \quad (\mathrm{i} = 3, 4, 5) \\
g_{5+2i+1}(\mathbf{x}) &= x_i - 45 \leq 0 \quad (\mathrm{i} = 3, 4, 5)
\end{aligned}
$$

KKT proximity measure is plotted in figure 10 with a single GA run. The KKT proximity measure smoothly reduced, eventually converging to zero close to the $550^{th}$ generation. The GA outputs converge to the optima at [78.0 33.0 29.995 45.0 36.775] where $g_1$, $g_6$, $g_7$, $g_9$, $g_{14}$ are active.

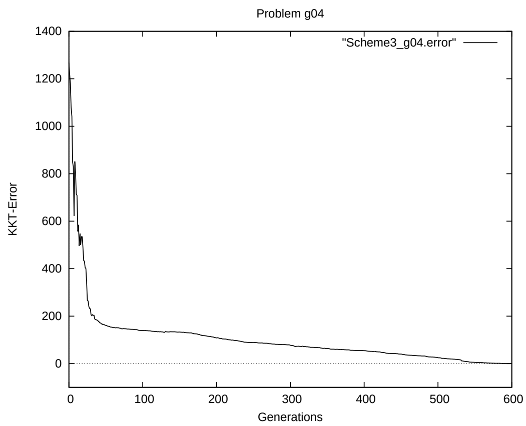



Figure 10: KKT-proximity measure for problem g04.

#### 6.4.6 Problem *g06*

The following is problem g06 from [13] contaning 2 variables, and 6 constraints, including the variable bounds.

Minimize:

$$f(\mathbf{x}) = (x_1 - 10)^3 + (x_2 - 20)^3$$

subject to:

$$\begin{aligned}
g_1(\mathbf{x}) &= -(x_1-5)^2 - (x_2-5)^2 + 100 \leq 0 \\
g_2(\mathbf{x}) &= (x_1-6)^2 + (x_2-5)^2 - 82.81 \leq 0 \\
g_3(\mathbf{x}) &= 13 - x_1 \leq 0 \\
g_4(\mathbf{x}) &= x_1 - 100 \leq 0 \\
g_5(\mathbf{x}) &= -x_2 \leq 0 \\
g_6(\mathbf{x}) &= x_2 - 100 \leq 0
\end{aligned}$$

The GA converges to the optimum in this problem (having $f^* = -6961.8139$), where we have found a zero KKT-proximity measure (figure 11). Constraints $g_1$ and $g_2$ are found to be active at the optimum. The multiplier values for active constraints are $u_1 = 1097.118894$ and $u_2 = 1229.542030$.

#### 6.4.7 Problem *g07*

The following is a 10 variable, 28 constraint problem g07 from [13].

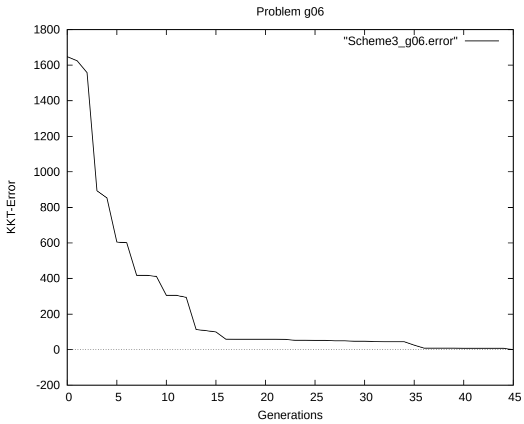



Figure 11: KKT-proximity measure for problem g06.

Minimize:

$$\begin{aligned} f(\mathbf{x}) &= x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 + \\ & \quad 4(x_4 - 5)^2 + (x_5 - 3)^2 + 2(x_6 - 1)^2 + 5x_7^2 + \\ & \quad 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45 \end{aligned}$$

subject to:

$$\begin{aligned} g_1(\mathbf{x}) &= -105 + 4x_1 + 5x_2 - 3x_7 + 9x_8 \leq 0 \\ g_2(\mathbf{x}) &= 10x_1 - 8x_2 - 17x_7 + 2x_8 \leq 0 \\ g_3(\mathbf{x}) &= -8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12 \leq 0 \\ g_4(\mathbf{x}) &= 3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 - 120 \leq 0 \\ g_5(\mathbf{x}) &= 5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40 \leq 0 \\ g_6(\mathbf{x}) &= x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_5 - 6x_6 \leq 0 \\ g_7(\mathbf{x}) &= 0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6 - 30 \leq 0 \\ g_8(\mathbf{x}) &= -3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10} \leq 0 \\ g_{7+2i}(\mathbf{x}) &= -10 - x_i \leq 0 \quad (\mathrm{i} = 1, \ldots, 10) \\ g_{7+2i+1}(\mathbf{x}) &= x_i - 10 \leq 0 \quad (\mathrm{i} = 1, \ldots, 10) \end{aligned}$$

The GA run converges within a Euclidean distance of 0.0298 from the reported optimum. At the manually appended optimum ($f^* = 24.3062$), the proximity measure obtained is zero (figure 12). Six constraints $\{g_1,\ g_2,\ g_3,\ g_4,\ g_5,\ g_6\}$ are active with corresponding multiplier values as $u_1 = 1.716531$, $u_2 = 0.474520$, $u_3 = 1.375921$, $u_4 = 0.020545$, $u_5 = 0.312027$ and $u_6 = 0.287049$.

### 6.4.8 Problem *g09*

The following problem, g09 from [13], is a 7 variable, 18 constraint problem.

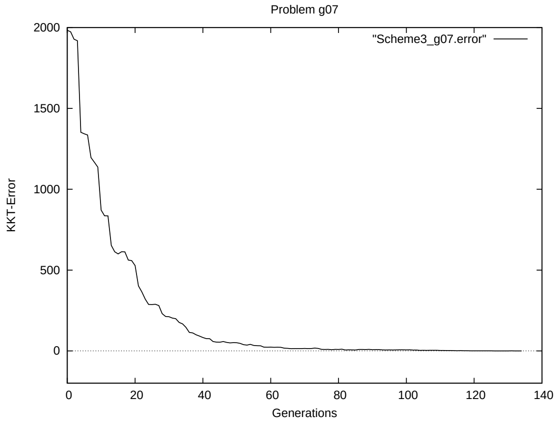



Figure 12: KKT-Proximity measure for problem g07.

Minimize:

$$f(\mathbf{x}) = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10x_5^6 + 7x_6^2 + x_7^4 - 4x_6x_7 - 10x_6 - 8x_7$$

subject to:

$$\begin{aligned} g_1(\mathbf{x}) &= -127 + 2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 \leq 0 \\ g_2(\mathbf{x}) &= -282 + 7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 \leq 0 \\ g_3(\mathbf{x}) &= -196 + 23x_1 + x_2^2 + 6x_6^2 - 8x_7 \leq 0 \\ g_4(\mathbf{x}) &= 4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7 \leq 0 \\ g_{3+2i}(\mathbf{x}) &= -10 - x_i \leq 0 \quad (\text{i} = 1, \ldots, 7) \\ g_{3+2i+1}(\mathbf{x}) &= x_i - 10 \leq 0 \quad (\text{i} = 1, \ldots, 7) \end{aligned}$$

The GA converges within a Euclidean distance of 0.0641 from the reported optimum. The KKT-proximity measure reduces with iterates, as shown in figure 13 and is zero at the manually appended optima. Two constraints $\{g_1, g_4\}$ are active at the reported solution with $u_1 = 1.139670$ and $u_4 = 0.368590$.

#### 6.4.9 Problem *g10*

The following problem, g10 from [13], is an 8 variable, 22 constraint problem. The objective function is affine, and constraints smooth.

Minimize:

$$f(\mathbf{x}) = x_1 + x_2 + x_3$$

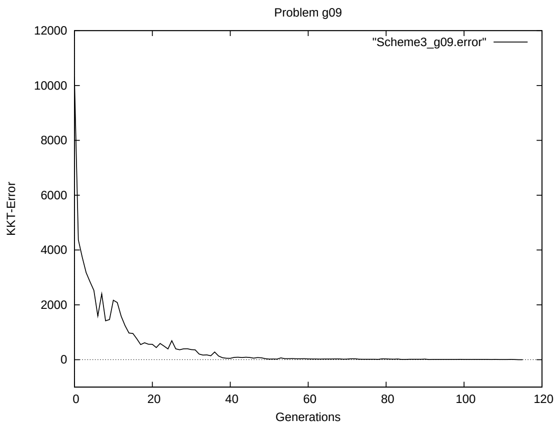



Figure 13: KKT-proximity measure for problem g09.

subject to:

$$
\begin{aligned}
g_1(\mathbf{x}) &= -1 + 0.0025(x_4 + x_6) \leq 0 \\
g_2(\mathbf{x}) &= -1 + 0.0025(x_5 + x_7 - x_4) \leq 0 \\
g_3(\mathbf{x}) &= -1 + 0.01(x_8 - x_5) \leq 0 \\
g_4(\mathbf{x}) &= -x_1x_6 + 833.33252x_4 \\
&\quad +100x_1 - 83333.333 \leq 0 \\
g_5(\mathbf{x}) &= -x_2x_7 + 1250x_5 + x_2x_4 - 1250x_4 \leq 0 \\
g_6(\mathbf{x}) &= -x_3x_8 + 1250000 + x_3x_5 - 2500x_5 \leq 0 \\
g_7(\mathbf{x}) &= 100 - x_1 \leq 0 \\
g_8(\mathbf{x}) &= x_1 - 10000 \leq 0 \\
g_9(\mathbf{x}) &= 1000 - x_2 \leq 0 \\
g_{10}(\mathbf{x}) &= x_2 - 10000 \leq 0 \\
g_{11}(\mathbf{x}) &= 1000 - x_3 \leq 0 \\
g_{12}(\mathbf{x}) &= x_3 - 10000 \leq 0 \\
g_{5+2i}(\mathbf{x}) &= 10 - x_i \leq 0 \quad (\text{i} = 4, \ldots, 8) \\
g_{5+2i+1}(\mathbf{x}) &= x_i - 1000 \leq 0 \quad (\text{i} = 4, \ldots, 8)
\end{aligned}
$$

The GA doesn't converge to the optimum in this problem. The best solution comes within an Euclidean distance of 1405.0 from the reported optimum. The proximity measure at the reported optimum (with $f^* = 7049.24802$) is found to be zero (Figure 14). All six constraints are active with multipliers as $u_1 = 1960.51566$, $u_2 = 5201.335892$, $u_3 = 5100.767947$, $u_4 = 0.008461$, $u_5 = 0.009561$ and $u_6 = 0.009982$, contrary to that in the previous study [13] which reported constraints $g_1$, $g_2$ and $g_3$ are active.

#### 6.4.10 Problem *g18*

The following problem, g18 from [13], is an 9 variable, 31 constraint problem.

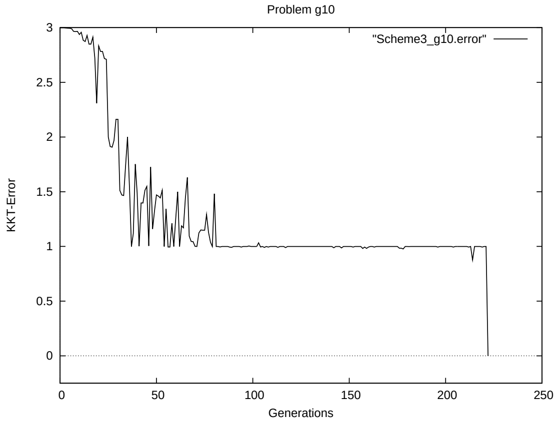



Figure 14: KKT-proximity measure for problem g10.

Minimize:

$$f(\mathbf{x}) = -0.5(x_1x_4 - x_2x_3 + x_3x_9 - x_5x_9 + x_5x_8 - x_6x_7)$$

subject to:

$$\begin{aligned}
g_1(\mathbf{x}) &= x_3^2 + x_4^2 - 1 \leq 0 \\
g_2(\mathbf{x}) &= x_9^2 - 1 \leq 0 \\
g_3(\mathbf{x}) &= x_5^2 + x_6^2 - 1 \leq 0 \\
g_4(\mathbf{x}) &= x_1^2 + (x_2 - x_9)^2 - 1 \leq 0 \\
g_5(\mathbf{x}) &= (x_1 - x_5)^2 + (x_2 - x_6)^2 - 1 \leq 0 \\
g_6(\mathbf{x}) &= (x_1 - x_7)^2 + (x_2 - x_8)^2 - 1 \leq 0 \\
g_7(\mathbf{x}) &= (x_3 - x_5)^2 + (x_4 - x_6)^2 - 1 \leq 0 \\
g_8(\mathbf{x}) &= (x_3 - x_7)^2 + (x_4 - x_8)^2 - 1 \leq 0 \\
g_9(\mathbf{x}) &= x_7^2 + (x_8 - x_9)^2 - 1 \leq 0 \\
g_{10}(\mathbf{x}) &= x_2x_3 - x_1x_4 \leq 0 \\
g_{11}(\mathbf{x}) &= -x_3x_9 \leq 0 \\
g_{12}(\mathbf{x}) &= x_5x_9 \leq 0 \\
g_{13}(\mathbf{x}) &= x_6x_7 - x_5x_8 \leq 0 \\
g_{12+2i}(\mathbf{x}) &= -10 - x_i \leq 0 \quad (\mathrm{i} = 1, \ldots, 8) \\
g_{12+2i+1}(\mathbf{x}) &= x_i - 10 \leq 0 \quad (\mathrm{i} = 1, \ldots, 8) \\
g_{30}(\mathbf{x}) &= -x_9 \leq 0 \\
g_{31}(\mathbf{x}) &= x_9 - 20 \leq 0
\end{aligned}$$

The GA run converges to the optima, and the KKT proximity measure is plotted in figure 15. After initial fluctuations, the KKT proximity measure converges to zero. At the optima, the constraints, $g_1$, $g_3$, $g_4$, $g_6$, $g_7$ and $g_9$ are active.

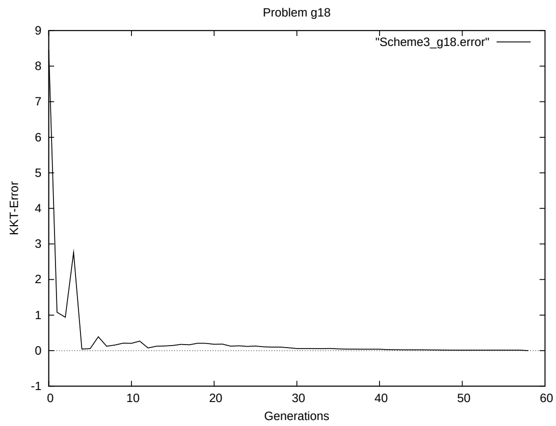



Figure 15: KKT-proximity measure for problem g18.

A summary of the problems, the optima, and the associated Langrangian multipliers of the active constraints are tabulated in Table 2.

# 7 Conclusions

This work is one of the few studies aimed at exploiting KKT conditions in optimization algorithm design. The KKT proximity measure proposed and tested provides not only a meaningful termination criterion but maybe integrated with random search techniques or classical algorithms to predict descent direction. This study also provides an insight into the nature of KKT conditions, and the need to relax the complimentary slackness conditions to arrive at a meaningful proximity measure. Simulations on the generation best outputs of a Real-Coded Genetic Algorithm support the suitability of our relaxation scheme.

Further work should be aimed at exploting the KKT proximity measure in heuristic algorithms and integration of the same with Evolutionary Computing as a means of local search.

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Table 2: Lagrange Multiplier Values At The Optimum Points.

| No. | Problem | Optima | Lagrange Multipliers |
|---|---|---|---|
| 1 | g01 | [ 1.0 1.0 1.0 1.0 1.0 1.0 1.0 1.0 1.0 3.0 3.0 3.0 1.0 ] | $u_2 = 0.49986960$, $u_3 = 0.49986960$, $u_4 = 0.49995217$ , $u_8 = 0.00062179$, $u_9 = 0.00036150$, $u_{10} = 0.00036151$, $u_{24} = 3.00034711$, $u_{25} = 3.00025284$, $u_{26} = 3.00025284$, $u_{27} = 5.00107578$, $u_{28} = 1.00055454$, $u_{29} = 1.00063755$, $u_{30} = 1.00031322$, $u_{31} = 1.00063756$, $u_{32} = 1.00031323$ , $u_{36} = 0.99998890$ |
| 2 | hs23 | [ 1.0 1.0 ] | $u_5 = 2.00046030$, $u_6 = 2.00060355$ |
| 3 | hs45 | [ 1.0 2.0 3.0 4.0 5.0 ] | $u_7 = 0.99951173$, $u_8 = 0.49975586$, $u_9 = 0.33317057$, $u_{10} = 0.24987793$, $u_{11} = 0.19990234$ |
| 4 | g02 | [ 3.16246061 3.128331428 3.09479212 3.061450595 3.0279292, 2.99382607, 2.95866871, 2.92184227, 0.494825115, 0.48835711, 0.4823164, 0.47664476, 0.4712955, 0.46623099, 0.46142, 0.4568366, 0.4524588, 0.448267622, 0.444247, 0.44038285 ] | $u_2 = 0.04689694$ |
| 5 | g04 | [ 78.0 33.0 29.995256 45.0 36.775813 ] | $u_2 = 403.27022000$, $u_7 = 809.42360424$, $u_8 = 48.92768703$, $u_10 = 84.32381214$, $u_{16} = 26.63967450$ |
| 6 | g06 | [ 14.095 0.842960789 ] | $u_2 = 1097.11096525$, $u_3 = 1229.53332532$, $u_5 = 0.00006220$ |
| 7 | g07 | [ 2.171996 2.363683 8.773926 5.095984 0.990655 1.430574 1.321644 9.828726 8.280092 8.375927 ] | $u_2 = 1.71648886$, $u_3 = 0.47450184$, $u_4 = 1.37590841$, $u_5 = 0.02054950$, $u_6 = 0.31202935$, $u_7 = 0.28707154$ |
| 8 | g09 | [ 2.3305084590 1.9513700830 -0.477418650 4.3657261380 -0.624486980 1.0381346830 1.5942188960 ] | $u_2 = 1.13972466$, $u_5 = 0.36850490$ |
| 9 | g10 | [ 579.306685 1359.970678 5109.970657 182.017699 295.601173 217.982300 286.416525 395.601173 ] | $u_2 = 1966.52920083$, $u_3 = 5217.30343e838$, $u_4 = 5116.48814974$, $u_5 = 0.00848649$, $u_6 = 0.00959083$, $u_7 = 0.01001275$ |
| 10 | g18 | [ -0.657776192 -0.153418773 0.323413872 -0.946257612 -0.657776192 -0.753213435 0.323413874 -0.346462948 0.599794663 ] | $u_2 = 0.14409510$, $u_4 = 0.14409508$, $u_5 = 0.14462060$, $u_7 = 0.14425899$, $u_8 = 0.14445993$, $u_{10} = 0.14408119$ |